(8)

Exercises 22.1.5

1. Show that if $\langle x_n \rangle_n = \langle c, c, c, \ldots \rangle$ is a constant sequence in $\mathbb{R}$, then $x_n \to c$.

for $x_n \to c$, we require that:

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geqslant N)[\,|x_n - c| < \varepsilon\,]$$

we assume $(\forall n)[\,x_n = c\,]$

then $(\forall n)[\,|x_n - c| = 0\,]$

then $(\forall \varepsilon > 0)(\forall n)[\,|x_n - c| > \varepsilon\,]$

then $(\forall \varepsilon > 0)(\forall N)(\forall n \geqslant N)[\,|x_n - c| < \varepsilon\,]$

so $(\forall \varepsilon > 0)(\exists N)(\forall n \geqslant N)[\,|x_n - c| < \varepsilon\,]$

and so by our definition, x_n is a convergent sequence with limit c, or $x_n \to c$.

2. Show that if $x_n \to x$ in $\mathbb{R}$, then $|x_n| \to |x|$

To show that $|x_n| \to |x|$, we must show that:

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)[\,||x_n| - |x|| < \varepsilon\,]$$

From the triangle inequality, we know that:

$$||x_n| - |x|| \leq |x_n - x|$$

So $||x_n - |x||$ is has upper bound $|x_n - x|$.

We have assumed that:

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)[\,|x_n - x| < \varepsilon\,]$$

If $|x_n - x| < \varepsilon$, then $||x_n| - |x|| < \varepsilon$ by transitivity of an ordered field.

Then, for any ε, we can find an N such that $(\forall n \geq N)$, $|x_n - x| < \varepsilon$ and $||x_n| - |x|| < \varepsilon$.

So we can conclude that $|x_n|$ is a convergent sequence with a limit $|x|$.

3. No, because the only possible elements of the seavence are zero or one. Hence, the only possible limits are zero or one.

Both of these are not within any $\varepsilon > 0$ of the other.

But, because it is cool, let's try proving the seavence does converge:

For the set $[x_n, x_{n-1}]$, there are $\frac{100}{n}\%$ 1s, and $(100 - \frac{100}{n})\%$ 0s. So these sets converge to a seavence of only zeros. A seavence of 0s converges to 0, and hence the seavence, in some way, converges to 0.